Journal of Engineering Mathematics, Vol. 13, No. 1, January 1979

One-dimensional plastic materials with work-hardening

I. Constitutive equations and stress-strain relations

T. TOKUOKA

Department of Aeronautical Engineering, Kyoto University, Kyoto, Japan

(Received March 23, 1978)

SUMMARY

A one-dimensional plastic material is proposed which shows isotropic and translational work-hardening. The tangential moduli of the stress and two internal state variables with respect to the strain are assumed to be functions of the stress and the internal state variables. Five constitutive assumptions are made and the resulting constitutive equations are similar to the equations of a three-dimensional rate-type plastic material in the case of uniaxial stress extension.

1. Introduction

Recently the author proposed a *rate-type plastic material* [1] which shows *general work-hardening*, that is, isotropic and translational work-hardening. The constitutive equation has the form of *hypo-elasticity* framed by Truesdell [2], and two *internal state variables*, scalar and tensor, are introduced for the sake of the endowment of the work-hardening. Internal state variables have been introduced and discussed by several investigators [3] – [8].

Our physical space is three-dimensional and all of the material has the characteristics due to the three spatial dimensions, and, usually, some complex phenomenon occurs due to the spatially plural dimension. On the other hand, in one-dimensional space, all of the quantities are numbers and we can easily analyse and estimate the intrinsic material properties without much loss of generality.

In this article we shall investigate a plastic material with general work-hardening in one-dimensional space and we shall compare its stress-strain relation with the response of a rate-type plastic material in three-dimensional space for uniaxial stress extension.

2. Definition of constitutive equations

A material particle which has the coordinate X in the reference configuration takes the place x in the current configuration at time t. The motion is expressed as

$$x = \chi(X, t). \tag{2.1}$$

The deformation gradient and the strain are defined, respectively, by

$$F = \frac{\partial x}{\partial X}, \qquad e = F - 1. \tag{2.2}$$

The stress T and the internal state variables α and β are assumed to be functions of the strain, and their tangential moduli with respect to the strain are functions of T, α and β. Thus,

$$\frac{dT}{de} = H(T, \alpha, \beta), \tag{2.3a}$$

$$\frac{d\alpha}{de} = \Phi(T, \alpha, \beta), \tag{2.3b}$$

$$\frac{d\beta}{de} = \Psi(T, \alpha, \beta). \tag{2.3c}$$

These are the constitutive equations of the plastic material. In the next section we will define the loading and the unloading state and we will assume that the tangential moduli have different forms in the two states.

From the equations we can obtain that*

$$\dot{T} = H(T, \alpha, \beta)\,\dot{e}, \tag{2.4a}$$

$$\dot{\alpha} = \Phi(T, \alpha, \beta), \tag{2.4b}$$

$$\dot{\beta} = \Psi(T, \alpha, \beta)\,\dot{e}, \tag{2.4c}$$

and

$$\frac{\partial T}{\partial X} = H(T, \alpha, \beta)\,\frac{\partial e}{\partial X}, \tag{2.5a}$$

$$\frac{\partial \alpha}{\partial X} = \Phi(T, \alpha, \beta)\,\frac{\partial e}{\partial X}, \tag{2.5b}$$

$$\frac{\partial \beta}{\partial X} = \Psi(T, \alpha, \beta)\,\frac{\partial e}{\partial X}. \tag{2.5c}$$

* The rate-type plasticity [1] has the constitutive equation

$$\mathring{\boldsymbol{T}} = \mathbf{H}(\boldsymbol{T}, \alpha, \beta)\,\boldsymbol{D},$$

where $\mathring{\boldsymbol{T}}$ is the corrotational time rate of stress $\boldsymbol{T}$ and $\boldsymbol{D}$ is the stretching. The one-dimensional expression of the stress is T but that of the stretching is $\partial\dot{x}/\partial x = \dot{e}/F \neq \dot{e}$. Then (2.3a) is not the one-dimensional expression of the constitutive equation of the three-dimensional rate-type plastic material, but (2.3) must be regarded as a new definition of the material.

The internal state variables α and β are called, respectively, the *parameter of the strain-hardening* and the *translation.* An increase of α means *isotropic work-hardening* and a variation of β means *translation of the center of the yield points.* The translated stress

$$\widetilde{T} \equiv T - \beta \tag{2.6}$$

is the relative stress when the origin of the one-dimensional stress space is translated to β.

3. Constitutive assumptions

In order to get more concrete expressions for the constitutive equations, we shall consider the following five constitutive assumptions.

(i) The tangential moduli H, Φ and Ψ depend on the stress and the translation through the translated stress. Thus, we have

$$\frac{dT}{de} = H(\widetilde{T}, \alpha), \tag{3.1a}$$

$$\frac{d\alpha}{de} = \Phi(\widetilde{T}, \alpha), \tag{3.1b}$$

$$\frac{d\beta}{de} = \Psi(\widetilde{T}, \alpha). \tag{3.1c}$$

From (3.1a) and (3.1c) we have

$$\frac{d\widetilde{T}}{de} = K(\widetilde{T}, \alpha), \tag{3.2}$$

where

$$K(\widetilde{T}, \alpha) = H(\widetilde{T}, \alpha) - \Psi(\widetilde{T}, \alpha). \tag{3.3}$$

Thus the set of the constitutive equations consists of (3.2), (3.1b) and (3.1c).

(ii) The translated stress makes the work

$$w = \widetilde{T}\,\frac{\dot{e}}{F} \tag{3.4}$$

per unit time. When $w > 0$, the work is done on the body by the exterior; when $w < 0$, the body makes the work on the exterior. Therefore we may define that the loading state is the state where the work is done on the body, that is, $w > 0$, and the unloading state is the state where the body makes the work on the exterior, that is, $w < 0$.

As we noted in the first section, the tangential moduli have different forms in the loading and the unloading state. We assume that in the unloading state

$$K = \lambda, \qquad \Phi = \Psi = 0, \tag{3.5}$$

where λ is a material constant, and in the loading state

$$\frac{d\widetilde{T}}{de} = \lambda + K_{\mathrm{P}}(\widetilde{T}, \alpha) \tag{3.6}$$

and K_{P}, Φ and Ψ are identically nonvanishing functions.

We assume also that at $\widetilde{T} = 0$ the difference between the loading and the unloading state disappears, that is,

$$K_{\mathrm{P}}(0, \alpha) = \Phi(0, \alpha) = \Psi(0, \alpha) = 0. \tag{3.7}$$

(iii) The moduli K_{P} and Ψ are even functions of $\widetilde{T}$ and the modulus Φ is an odd function of it:

$$K_{\mathrm{P}}(-\widetilde{T}, \alpha) = K_{\mathrm{P}}(\widetilde{T}, \alpha), \qquad \Phi(-\widetilde{T}, \alpha) = -\Phi(\widetilde{T}, \alpha), \qquad \Psi(-\widetilde{T}, \alpha) = \Psi(\widetilde{T}, \alpha). \tag{3.8}$$

These symmetry relations indicate that $\dot{\widetilde{T}} - \lambda\dot{e}$ and $\dot{\beta}$ have opposite signs at $\widetilde{T} > 0$ and $\widetilde{T} < 0$, and $\dot{\alpha}$ has the same sign, because the loading-state condition $w > 0$ results in $\dot{e} > 0$ and $\dot{e} < 0$ according to $\widetilde{T} > 0$ and $\widetilde{T} < 0$, respectively. If we take $\dot{\alpha} = \Phi(\widetilde{T}, \alpha)\dot{e} > 0$, the parameter of the strain-hardening α increases monotonically.

(iv) The tangential moduli K_{P}, Φ and Ψ are assumed to be quadratic polynomials in $\widetilde{T}$. Then, from the assumptions (ii), (iii) and (iv) we have

$$K_{\mathrm{P}}(\widetilde{T}, \alpha) = k(\alpha)\widetilde{T}^2, \qquad \Phi(\widetilde{T}, \alpha) = \phi(\alpha)\widetilde{T}, \qquad \Psi(\widetilde{T}, \alpha) = \psi(\alpha)\widetilde{T}^2, \tag{3.9}$$

where $k(\alpha)$, $\phi(\alpha)$ and $\psi(\alpha)$ are material functions of α.

From the monotonic property of α, the quantity

$$\alpha' = \frac{1}{\lambda}\int_0^{\alpha} \frac{d\alpha}{\phi(\alpha)} \tag{3.10}$$

can be defined uniquely, when a material is assigned. So equation (3.1b) in the loading state reduces to

$$\frac{d\alpha}{de} = \frac{\widetilde{T}}{\lambda}, \tag{3.11}$$

where α' defined in (3.10) is rewritten as α. From now on we adopt the new quantity (3.10) as the parameter of strain-hardening.

(v) The two material functions $k(\alpha)$ and $\psi(\alpha)$ are assumed to be proportional to each other, and we put

$$k(\alpha) = -\frac{1}{c}\psi(\alpha) = -\frac{1}{\lambda M(\alpha)^2}, \tag{3.12}$$

where $M(\alpha)$ is a new positive material function and c is a material constant. Then the two constitutive equations (3.2) and (3.1c) in the loading state are given by

$$\frac{d\widetilde{T}}{de} = \lambda - \frac{\widetilde{T}^2}{\lambda M(\alpha)^2}, \tag{3.13}$$

$$\frac{d\beta}{de} = \frac{c\widetilde{T}^2}{\lambda M(\alpha)^2}. \tag{3.14}$$

4. Yield condition

The *yield condition* is defined by the condition that the constitutive equation (3.13) in the loading state has vanishing tangent modulus, that is

$$\widetilde{T} = \pm\lambda M(\alpha). \tag{4.1}$$

When the above condition is satisfied, $d\widetilde{T}/de = 0$ holds and it might be supposed that an indefinite elongation occurs at a constant stress. But this is not correct, because the variation of the strain gives rise to a variation of α by equation (3.11) unless the stress vanishes, so the variation of $M(\alpha)$ breaks up the yield condition (4.1) if the stress remains constant.

Let us now estimate the *yield translated stress.* When the material function $M(\alpha)$ is identically constant, relation (4.1) holds exactly. When $M(\alpha)$ is not a constant, (4.1) can not hold but it can be regarded as the first-order approximation of the yield value of the translated stress. Then we can obtain

$$\frac{d\widetilde{T}}{de} \cong \pm\lambda\frac{dM(\alpha)}{de} = \pm\widetilde{T}M(\alpha)' \cong \lambda M(\alpha)M(\alpha)' \tag{4.2}$$

where $M(\alpha)' = dM(\alpha)/d\alpha$ and (3.11) is used. Substituting (4.2) into (3.13) we have the second-order approximation of the yield value of the translated stress

$$\widetilde{T} = \pm\lambda M(\alpha)(1 - M(\alpha)M(\alpha)')^{\frac{1}{2}}. \tag{4.3}$$

If the function $M(\alpha)$ is a monotonically increasing function of α, that is, $M(\alpha)' > 0$, the second-order value (4.3) is less that the first-order value (4.1). Using (4.3) instead of (4.1) we can obtain the third-order approximation.

As regards α, the translated stress (4.3) gives rise to the variation of it. This phenomenon indicates *isotropic work-hardening.*

From (4.3) we have the yield stress

$$T = \beta \pm \lambda M(\alpha)\,(1 - M(\alpha)\,M(\alpha)')^{\frac{1}{2}}\,. \tag{4.4}$$

Then, by the constitutive equation (3.14), variation of the strain yields a variation of the translation β and of the stress. This phenomenon indicates translational work-hardening.

Figure 1 depicts the one-dimensional stress space. O is the origin, C is the translated center and $\mathrm{O\,C} = \beta$, Y_- and Y_+ denote the yield points and $\mathrm{Y}_-\mathrm{C} = \mathrm{C\,Y}_+$ is the yield value of the translated stress. The movement of C indicates the translational work-hardening and the spread of the width $\mathrm{Y}_-\mathrm{Y}_+$ indicates the isotropic work-hardening.

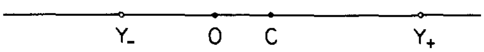

Figure 1. One-dimensional stress space, where C is the translation point and $\mathrm{Y}_\pm$ are the yield points.

5. Stress-strain relations

Let us write down non-dimensional expressions for the constitutive equations. The non-dimensional forms of the stress, the translational and the translated stress are defined, respectively, by

$$S = \frac{T}{\lambda}, \qquad \gamma = \frac{\beta}{\lambda}, \qquad \tilde{S} = \frac{\tilde{T}}{\lambda} = S - \gamma. \tag{5.1}$$

The quantities α and e, the material function $M(\alpha)$ and the material constant c are non-dimensional.

The constitutive equations* in the unloading state are

$$\frac{d\tilde{S}}{de} = 1, \qquad \frac{d\alpha}{de} = 0, \qquad \frac{d\gamma}{de} = 0, \tag{5.2}$$

* When we replace the quantities by the respective starred ones,

$$\tilde{S}^* = \frac{3}{2}\,\tilde{S}, \qquad \alpha^* = \frac{3}{2}\,\alpha, \qquad \gamma^* = \frac{3}{2}\,\gamma, \qquad e^* = e,$$

$$M^*(\alpha^*) = \sqrt{\frac{3}{2}}\,M(\alpha), \qquad c^* = c,$$

the constitutive equations (5.2) and (5.3) are those of a three-dimensional rate-type plastic material in the case of uniaxial stress extension, where e^* must be regarded as the logarithmic strain.

and those in the loading state are

$$\frac{d\tilde{S}}{de} = 1 - \frac{\tilde{S}^2}{M(\alpha)^2}, \qquad \frac{d\alpha}{de} = \tilde{S}, \qquad \frac{d\gamma}{de} = \frac{c\tilde{S}^2}{M(\alpha)^2}. \tag{5.3}$$

All of the one-dimensional continuum must satisfy the equation of motion

$$\frac{\partial T}{\partial X} + \rho_R\, b = \rho_R \ddot{x}, \tag{5.4}$$

where $\ddot{x}$ is the acceleration, ρ_R is the mass density, b is the body force and ρ_R and b refer to the reference configuration.

In the case of homogeneous stress, the accelerationless motion of a body with no body force satisfies the equation of motion automatically. Let us assume that the initial state is vanishing strain with respect to the homogeneous reference configuration and that the strain is homogeneous in the motion. So the stress will be homogeneous everywhere, (5.4) is satisfied by the accelerationless motion with no body force.

In the unloading state we can obtain

$$\tilde{S} = \tilde{S}_0 + e - e_0, \qquad \alpha = \alpha_0, \qquad \gamma = \gamma_0, \tag{5.5}$$

where the quantities provided with subscript zero denote the initial values of the respective quantities.

In the loading state we can easily solve the differential equations (5.3), and we have

$$\tilde{S}(\alpha, \alpha_0) = \pm \frac{1}{f(\alpha, \alpha_0)} \{\tilde{S}_0{}^2 + 2 \int_{\alpha_0}^{\alpha} f(\xi, \alpha_0)^2\, d\xi\}^{\frac{1}{2}}, \tag{5.6a}$$

$$e(\alpha, \alpha_0) = e_0 + \int_{\alpha_0}^{\alpha} \frac{d\xi}{\tilde{S}(\xi, \alpha_0)}, \tag{5.6b}$$

$$\gamma(\alpha, \alpha_0) = \gamma_0 + c \int_{\alpha_0}^{\alpha} \frac{\tilde{S}(\xi, \alpha_0)^2}{M(\xi)^2}\, d\xi, \tag{5.6c}$$

where

$$f(\alpha, \alpha_0) = \exp\left(\int_{\alpha_0}^{\alpha} \frac{d\xi}{M(\xi)^2}\right). \tag{5.7}$$

The stress can be evaluated as

$$S(\alpha, \alpha_0) = \tilde{S}(\alpha, \alpha_0) + \gamma(\alpha, \alpha_0). \tag{5.8}$$

Figures 2 show the stress, the translation and the parameter of the isotropic work-hardening plotted as functions of the strain. Here the material function is assumed to be

$$M(\alpha) = M_0(1 + a\alpha)^n, \tag{5.9}$$

where $M_0 > 0$, $a \geqslant 0$, $n \geqslant 0$ are non-dimensional material constants and $M_0 = \sqrt{1.5} \times 10^{-3}$ is presumed. The materials are loaded from the initial state $S = \alpha = \gamma = 0$ to positive and negative strain directions. They are unloaded from $e = 10^{-2}$ to the state $S = 0$, and are elongated or contracted again. Figure (a) refers to a perfectly plastic material prescribed by $a = c = 0$, Figure (b) to a material with isotropic work-hardening, Figures (c) and (d) to materials with translational work-hardening, and Figures (e) and (f) to materials with general work-hardening. The figures depict stress – strain diagrams (real bold line), translation – strain diagrams (real fine line), and parameter of the isotropic work-hardening – strain diagrams (broken line), where a black circle and a white circle denote, respectively, the starting state of the loading and the unloading.

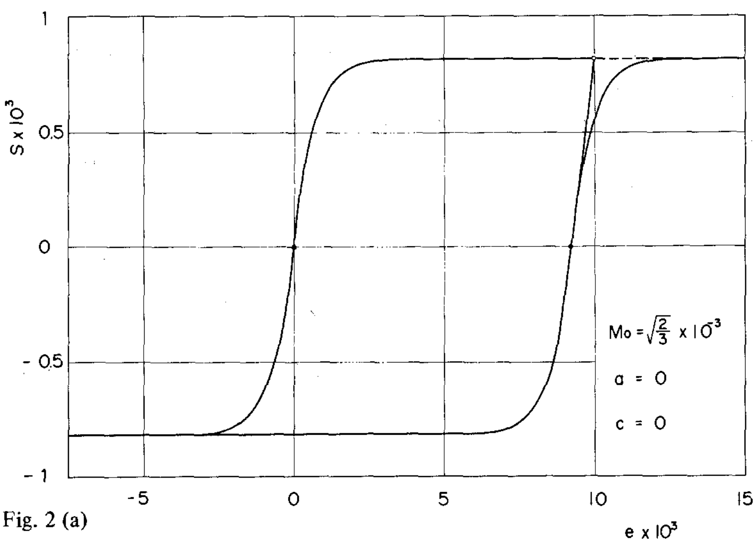

Fig. 2 (a)

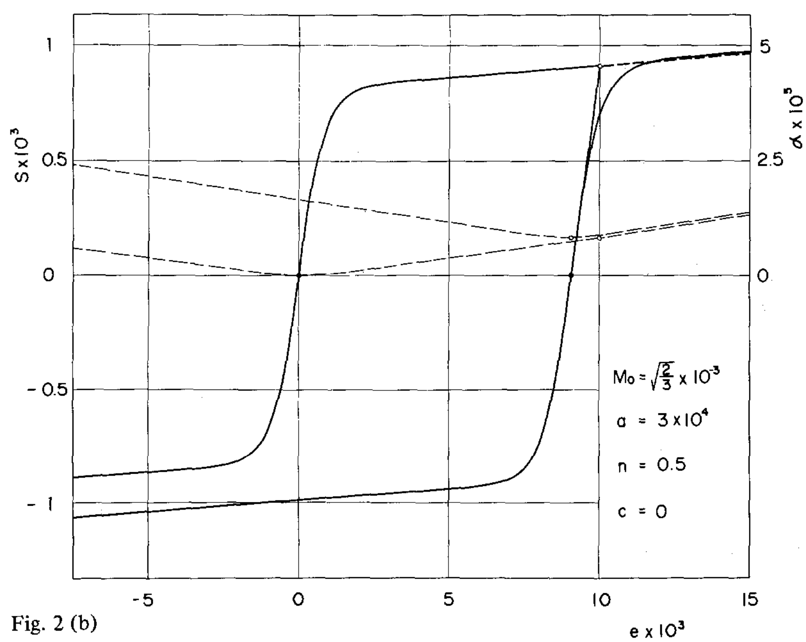

Fig. 2 (b)

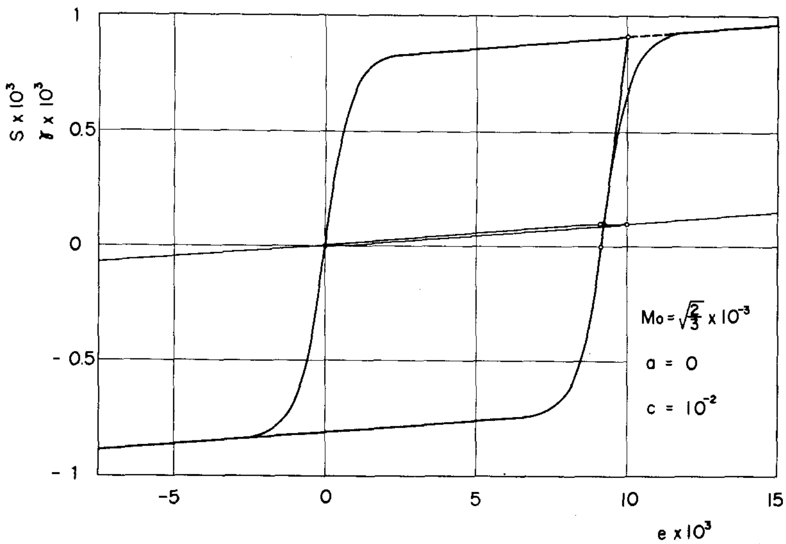
$S \times 10^3$
$\gamma \times 10^3$
$M_0 = \sqrt{\frac{2}{3}} \times 10^{-3}$
$a = 0$
$c = 10^{-2}$
$e \times 10^3$

Fig. 2 (c)

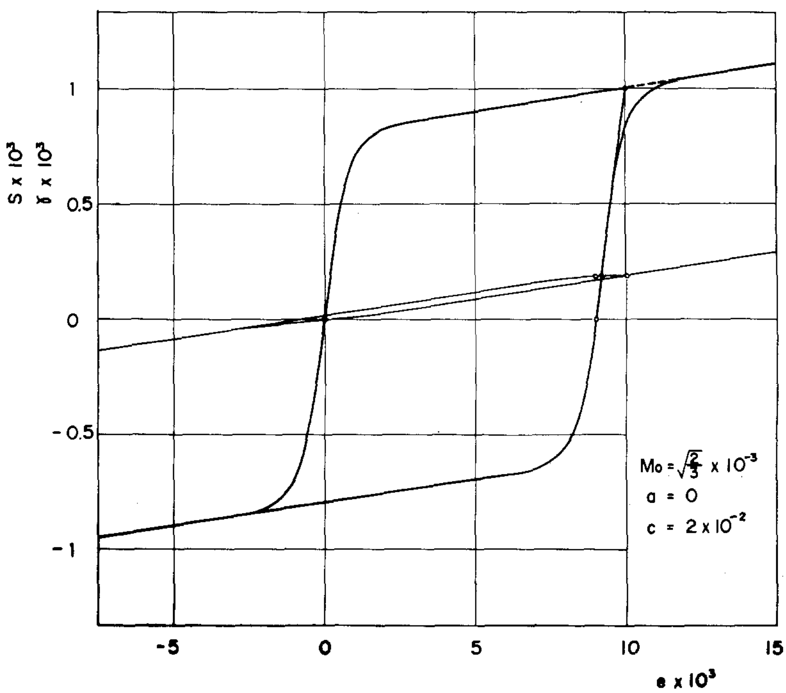
$S \times 10^3$
$\gamma \times 10^3$
$M_0 = \sqrt{\frac{2}{3}} \times 10^{-3}$
$a = 0$
$c = 2 \times 10^{-2}$
$e \times 10^3$

Fig. 2 (d)

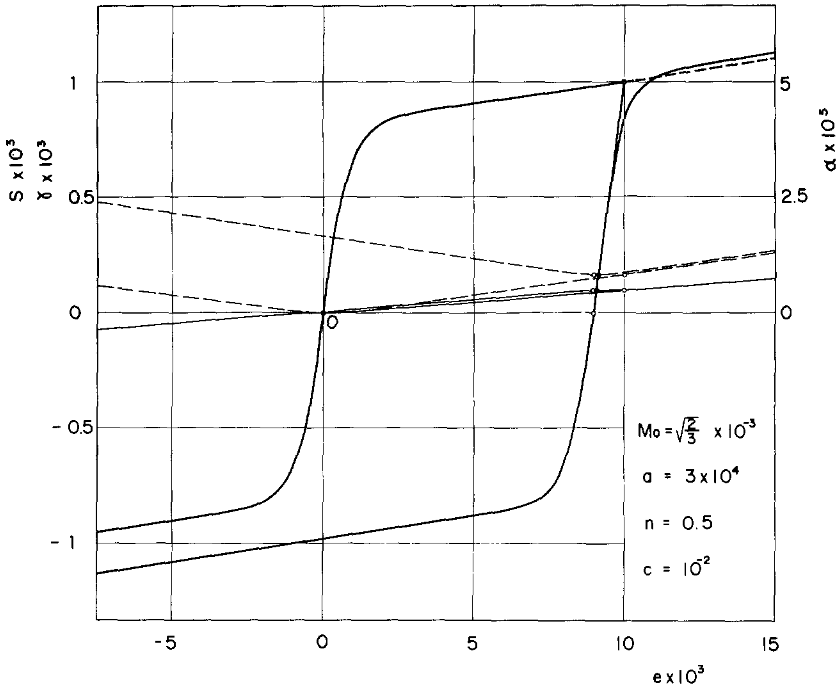
S x 10³
γ x 10³
α x 10⁵
e x 10³
Mo = √(2/3) x 10⁻³
a = 3 x 10⁴
n = 0.5
c = 10⁻²

Fig. 2 (e)

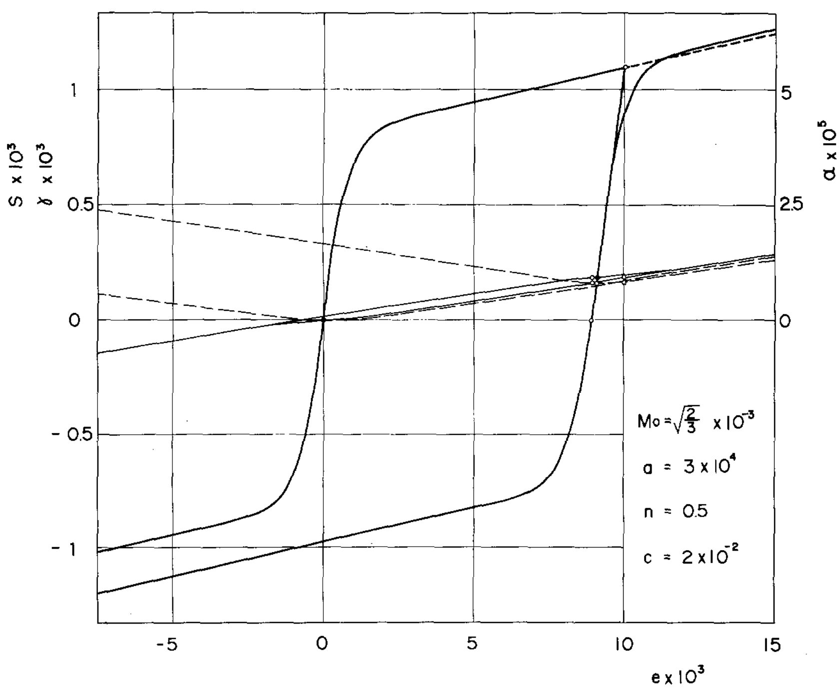
S x 10³
γ x 10³
α x 10⁵
e x 10³
Mo = √(2/3) x 10⁻³
a = 3 x 10⁴
n = 0.5
c = 2 x 10⁻²

Fig. 2 (f)

REFERENCES

[1] T. Tokuoka, Rate-type plastic material with general work-hardening, *Zeit. angew. Math. Mech.* 58 (1978)

[2] C. Truesdell, Hypo-elasticity, *J. Rational Mech. Anal.* 4 (1955) 83-133, 1019-1020.

[3] C. Truesdell and R. Toupin, *The classical field theories*, Handbuch der Physik Vol. III/1, Ed. by S. Flügge, Springer-Verlag, Berlin/Göttingen/Heidelberg (1960) 615-647.

[4] K. C. Valanis, Thermodynamics of large viscoelastic deformations, *J. Math. Phys.* 45 (1966) 197-212.

[5] B. D. Coleman and M. E. Gurtin, Thermodynamics with internal state variables, *J. Chem. Phys.* 47 (1967) 597-613.

[6] P. Perzyna, Thermodynamic theory of viscoplasticity, *Advances in Applied Mechanics*, Ed. by C.-S. Yih, Academic Press,, New York/London (1971) 313-354.

[7] J. Lubliner, On the structure of the rate equations of materials with internal variables, *Act. Mech.* 17 (1973) 109-119.

[8] S. Nemat-Nasser, On nonlinear thermoelasticity and nonequilibrium thermodynamics, in: *Nonlinear elasticity*, Ed. by R. W. Dickey, Academic Press, New York/London (1973) 289-338.